International Journal of Engineering Works
Kambohwell Publisher Enterprises
ISSN: 2409-2770 Vol. 3, Issue 10, PP. 73-77, October 2016
www.kwpublisher.com

On the Hausdorff Distance Between the Shifted Heaviside Function and Some Generic Growth Functions

Nikolay Kyurkchiev and Anton Iliev

***Abstract*—** In this paper we study the one–sided Hausdorff distance between the shifted Heaviside function and some generic growth function such as Turner–Bradley–Kirk–Pruitt function. Numerical examples are presented using *CAS MATHEMATICA*.

***Keywords*—** Sigmoid functions, Heaviside function, Turner–Bradley–Kirk–Pruitt generic function, Hausdorff distance, Upper and lower bounds

I. Introduction

We study the one–sided Hausdorff approximation of the shifted Heaviside function by Turner–Bradley–Kirk–Pruitt generic growth function. Precise upper and lower bounds for the Hausdorff distance have been obtained.

The estimates obtained give more insight on the lag phase, growth phase and plateau phase in the growth process [1]–[4].

In [5] we study the uniform approximation of the cut function by smooth sigmoidal generic logistic functions such as Nelder [6] and Turner–Blumenstein–Sebaugh [7].

II. The Shifted Heaviside Function and the Turner–Bradley–Kirk–Pruitt Generic Growth Function

Definition. *The shifted Heaviside function* $h_{t^*}(t)$ *is defined for* $t \in \mathbb{R}$ *by*

$$h_{t^*}(t) = \begin{cases} 0, & \text{if} \quad t < t^*, \\ [0,1], & \text{if} \quad t = t^*, \\ 1, & \text{if} \quad t > t^*. \end{cases} \qquad (1)$$

Definition. *The Hausdorff distance* $\rho(f,g)$ *between two interval functions* f,g *on* $\Omega \subseteq \mathbb{R}$, *is the distance between their completed graphs* $F(f)$ *and* $F(g)$ *considered as closed subsets of* $\Omega \times \mathbb{R}$ *[8], [9], [10].*

More precisely, we have

$$\rho(f,g) = \max\{ \sup_{A \in F(f)} \inf_{B \in F(g)} \|A - B\|, \sup_{B \in F(g)} \inf_{A \in F(f)} \|A - B\|\}, \qquad (2)$$

wherein $\|.\|$ *is any norm in* $\mathbb{R}^2$, *e. g. the maximum norm* $\|(t,x)\| = \max\{|t|,|x|\}$; *hence the distance between the points* $A = (t_A, x_A)$, $B = (t_B, x_B)$ *in* $\mathbb{R}^2$ *is* $\|A - B\| = \max(|t_A - t_B|, |x_A - x_B|)$.

In 1976 Turner, Bradley, Kirk and Pruitt [11] proposed a modified Verhulst logistic equation [12], [29] which they termed the *generic growth function*.

This has the form

$$\frac{dN}{dt} = rN^{1+\beta(1-\gamma)} \left(1 - \left(\frac{N}{K}\right)^{\beta}\right)^{\gamma}, \qquad (3)$$

where β, γ are positive parameters and $\gamma < 1 + \frac{1}{\beta}$.

This has the solution [13]:

$$N(t) = \frac{K}{\left(1 + \left((\gamma - 1)\beta r K^{\beta(1-\gamma)} t + \left(\left(\frac{K}{N0}\right)^{\beta} - 1\right)^{1-\gamma}\right)^{\frac{1}{1-\gamma}}\right)^{\frac{1}{\beta}}}, \qquad (4)$$

Note that the population at the inflection point, is given by

$$N_{inflection} = \left(1 - \frac{\beta\gamma}{1+\beta}\right)^{\frac{1}{\beta}} K. \qquad (5)$$

Nikolay Kyurkchiev, nkyurk@math.bas.bg, Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Acad. G. Bonchev Str., Bl. 8, 1113 Sofia, Bulgaria
Anton Iliev, aii@uni-plovdiv.bg, Faculty of Mathematics and Informatics, Paisii Hilendarski University of Plovdiv, 24 Tsar Assen Str., 4000 Plovdiv, Bulgaria

The condition $\gamma < 1 + \frac{1}{\beta}$ ensures that $N_{inflection} > 0$.

Special case. Let $K = 1$, $r = 1$ and

$$A = (\gamma - 1)\beta ; \quad B = \left(\left(\frac{1}{N0} \right)^{\beta} - 1 \right)^{1-\gamma}.$$

Then we obtain the special Turner–Bradley–Kirk–Pruitt function:

$$N(t) = \frac{1}{\left(1 + \left(At + B \right)^{\frac{1}{1-\gamma}} \right)^{\frac{1}{\beta}}}. \tag{6}$$

The function defined by (6) has an inflection at point $(t^*, N(t^*))$. Let us choose (see (5))

$$N(t^*) = \left(1 - \frac{\beta\gamma}{1+\beta} \right)^{\frac{1}{\beta}} = \frac{1}{2}. \tag{7}$$

Then

$$\gamma = \left(\frac{1+\beta}{\beta} \right) \left(1 - \left(\frac{1}{2} \right)^{\beta} \right). \tag{8}$$

On the other hand,

$$N(t^*) = \frac{1}{\left(1 + \left(At^* + B \right)^{\frac{1}{1-\gamma}} \right)^{\frac{1}{\beta}}} = \frac{1}{2}$$

and we find

$$t^* = \frac{\left(2^{\beta} - 1 \right)^{1-\gamma} - B}{A}.$$

III. Approximation of the shifted Heaviside function by Turner–Bradley–Kirk–Pruitt generic growth function (6)

We next focus on the one-sided Hausdorff approximation d of the function $h_{t^*}(t)$ by generic growth function (6).

Let

$$a = \frac{1}{2}$$

$$b = 1 - A\frac{\left(2^{\beta} - 1 \right)^{\gamma}}{\beta(1-\gamma)2^{\beta+1}} \tag{9}$$

The following Theorem gives upper and lower bounds for d

Theorem 3.1 *For the Hausdorff distance d between the function $h_{t^*}(t)$ and the function (6) the following inequalities hold for $\frac{A(2^{\beta}-1)^{\gamma}}{\beta(\gamma-1)2^{\beta+1}} > \frac{1}{2}$:*

$$d_l = \frac{1}{3\left(1 - A\frac{(2^{\beta}-1)^{\gamma}}{\beta(1-\gamma)2^{\beta+1}} \right)} < d <$$

$$\frac{\ln\left(3\left(1 - A\frac{(2^{\beta}-1)^{\gamma}}{\beta(1-\gamma)2^{\beta+1}} \right) \right)}{3\left(1 - A\frac{(2^{\beta}-1)^{\gamma}}{\beta(1-\gamma)2^{\beta+1}} \right)} = d_r. \tag{10}$$

Proof. We need to express d in terms of γ and β.

The Hausdorff distance d satisfies the relation

$$F(d) := N(t^* - d) = \frac{1}{\left(1 + \left(A(t^* - d) + B \right)^{\frac{1}{1-\gamma}} \right)^{\frac{1}{\beta}}} = d \tag{11}$$

Consider the function $G(d) = a - bd$.

By means of Taylor expansion we obtain

$$G(d) - F(d) = O(d^2).$$

Hence $G(d)$ approximates $F(d)$ with $d \to 0$ as $O(d^2)$ (see Fig. 3).

Further, for $\frac{A(2^{\beta}-1)^{\gamma}}{\beta(\gamma-1)2^{\beta+1}} > \frac{1}{2}$ we have

$$G(d_l) = \frac{1}{2} - \frac{1}{3} > 0,$$

$$G(d_r) = \frac{1}{2} - \frac{1}{3}\ln\left(3\left(1 - A\frac{(2^{\beta}-1)^{\gamma}}{\beta(1-\gamma)2^{\beta+1}} \right) \right) < 0.$$

This completes the proof of the theorem.

ISSN: 2409-2770

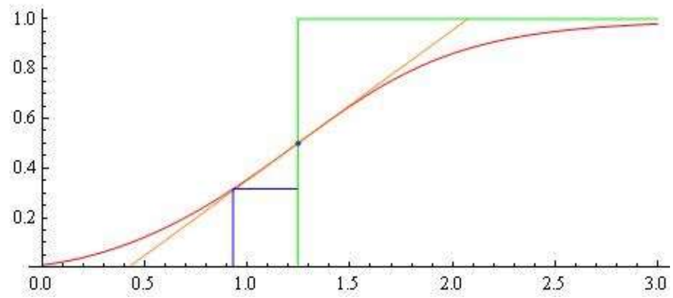

Figure 1: Approximation of the Heaviside function (green) with jump at point $t^* = 1.24604$ by the function $N(t)$ (red) with $K = 1$, $r = 1$, $\beta = 3$, $\gamma = 1.16667$, $N0 = 0.01$, Hausdorff distance $d = 0.315908$.

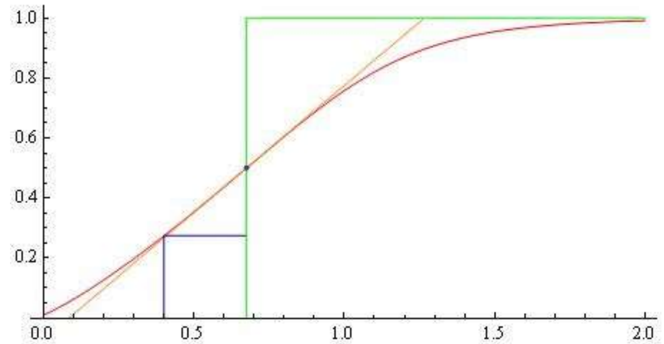

Figure 2: Approximation of the Heaviside function (green) with jump at point $t^* = 0.675232$ by the function $N(t)$ (red) with $K = 1$, $r = 1$, $\beta = 5$, $\gamma = 1.1625$, $N0 = 0.01$, Hausdorff distance $d = 0.274188$.

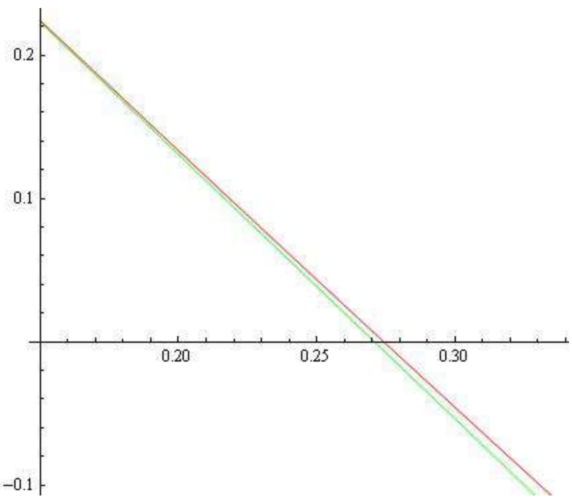

Figure 3: The functions $F(d)$ and $G(d)$ for $K = 1$, $r = 1$, $\beta = 3$, $\gamma = 1.16667$, $N0 = 0.01$.

Some comparison of the sigmoidal curve (6) and Verhulst logistic function

$$V(t) = \frac{1}{1 + e^{-k(t-t^*)}}$$

is plotted in Fig. 4.

For some approximation, computational and modelling aspects, see [14]–[26].

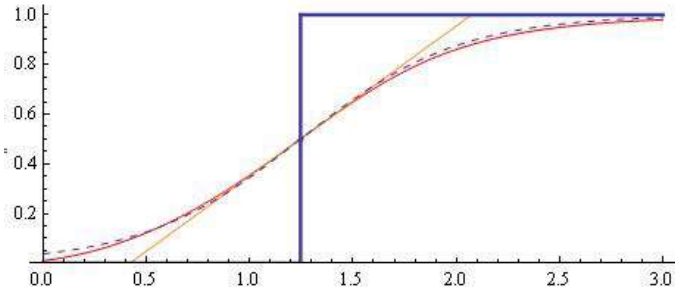

Figure 4: Approximation of the Heaviside function (thick) with jump at point $t^* = 1.24604$ by the function $N(t)$ (red) with $K = 1$, $r = 1$, $\beta = 3$, $\gamma = 1.16667$, $N0 = 0.01$ and by shifted Verhulst logistic function (dashed) with $k = 2.6$.

Remark 1. Blumberg [27] introduced the so called the *hyper–logistic function*:

$$\frac{dN}{dt} = rN^{\alpha}\left(1 - \frac{N}{K}\right)^{\gamma}. \tag{12}$$

The equation (12) is consistent with the Turner–Bradley–Kirk–Pruitt generic growth function (3) when $\beta = 1$, $\alpha = 2 - \gamma$ and $\gamma < 2$. Eq. (12) can be re–formulated as the integral equation (see, also [13])

$$\int_{\frac{N0}{K}}^{\frac{N(t)}{K}} x^{-\alpha}(1 - x)^{-\gamma}\,dx = rK^{\alpha-1}t.$$

Remark 2. The growth rate modelled by Gompertz function [28] is given by:

$$\frac{dN}{dt} = rN\left(\ln\left(\frac{K}{N}\right)\right)^{\gamma}. \tag{13}$$

With $\gamma > 0$, $\gamma \neq 1$, this special case is more usually known as the *hyper–Gompertz [11], generalized ecological growth function, or simply generalized Gompertz function*.

Equation (13) has the solution

$$N(t) = K\, exp\left(\left(\ln\left(\frac{N0}{K} \right) \right)^{1-\gamma} + r'(-1)^{\gamma}(1-\gamma).t \right)^{\frac{1}{1-\gamma}} . \qquad (14)$$

Based on the methodology proposed in the present note, the reader may formulate the corresponding approximation problems on his/her own.

REFERENCES

[1] S. Shoffner, S. Schnell, Estimation of the lag time in a subsequent monomer addition model for fibril elongation, bioRxiv The preprint server for biology (2015), 1–8, doi:10.1101/034900

[2] P. Arosio, T. P. J. Knowles, S. Linse, On the lag phase in amyloid fibril formation, Physical Chemistry Chemical Physics 17 (2015) 7606–7618, doi:10.1039/C4CP05563B

[3] N. Kyurkchiev, A note on the new geometric representation for the parameters in the fibril elongation process, Compt. rend. Acad. bulg. Sci. 69 (8) (2016) 963–972.

[4] S. Markov, Building reaction kinetic models for amyloid fibril growth, BIOMATH 5 2016, doi:10.11145/j.biomath.2016.07.311

[5] N. Kyurkchiev, S. Markov, Approximation of the cut function by some generic logistic functions and applications, Advances in Applied Sciences (2016) (accepted).

[6] J. A. Nelder, The fitting of a generalization of the logistic curve, Biometrics 17 (1961) 89–110.

[7] M. Turner, B. Blumenstein, J. Sebaugh, A Generalization of the Logistic Law of Growth, Biometrics 25 (3) (1969) 577–580.

[8] F. Hausdorff, Set Theory (2 ed.) (Chelsea Publ., New York, (1962 [1957]) (Republished by AMS-Chelsea 2005), ISBN: 978–0–821–83835–8.

[9] B. Sendov, Hausdorff Approximations (Kluwer, Boston, 1990), doi:10.1007/978-94-009-0673-0

[10] R. Anguelov, S. Markov, Hausdorff Continuous Interval Functions and Approximations, In: M. Nehmeier et al. (Eds), Scientific Computing, Computer Arithmetic, and Validated Numerics, 16th International Symposium, SCAN 2014, LNCS 9553 (2016) 3–13, Springer, doi:10.1007/978-3-319-31769-4

[11] M. Turner, E. Bradley, K. Kirk, K. Pruitt, A theory of growth, Math. Biosci. 29 (1976) 367–373.

[12] P. F. Verhulst, Notice sur la loi que la population poursuit dans son accroissement, Correspondance mathematique et physique 10 (1838) 113–121.

[13] A. Tsoularis, Analysis of logistic growth models, Les. Lett. Inf. Math. Sci. 2 (2001) 23–46.

[14] N. Kyurkchiev, S. Markov, On the Hausdorff distance between the Heaviside step function and Verhulst logistic function, J. Math. Chem. 54(1) (2016) 109–119, doi:10.1007/S10910-015-0552-0

[15] N. Kyurkchiev, S. Markov, Sigmoidal functions: some computational and modelling aspects, Biomath Communications 1 (2) (2014) 30–48; doi:10.11145/j.bmc.2015.03.081.

[16] A. Iliev, N. Kyurkchiev, S. Markov, On the approximation of the cut and step functions by logistic and Gompertz functions, Biomath 4 (2) (2015) 2–13.

[17] N. Kyurkchiev, S. Markov, On the approximation of the generalized cut function of degree by smooth sigmoid functions, Serdica J. Computing 9(1) (2015) 101–112.

[18] A. Iliev, N. Kyurkchiev, S. Markov, On the Approximation of the step function by some sigmoid functions, Mathematics and Computers in Simulation (2015), doi:10.1016/j.matcom.2015.11.005

[19] N. Kyurkchiev, A. Iliev, On some growth curve modeling: approximation theory and applications, Int. J. of Trends in Research and Development, 3(3) (2016) 466–471.

[20] N. Kyurkchiev, S. Markov, Sigmoid functions: Some Approximation and Modelling Aspects, LAP LAMBERT Academic Publishing, Saarbrucken (2015), ISBN: 978-3-659-76045-7.

[21] N. Kyurkchiev, A. Iliev, A note on some growth curves arising from Box-Cox transformation, Int. J. of Engineering Works 3(6) (2016) 47–51.

[22] N. Kyurkchiev, S. Markov, A. Iliev, A note on the Schnute growth model, Int. J. of Engineering Research and Development 12(6) (2016) 47–54.

[23] A. Iliev, N. Kyurkchiev, S. Markov, On the Hausdorff distance between the shifted Heaviside step function and the transmuted Stannard growth function, BIOMATH (2016) (accepted).

[24] N. Kyurkchiev, On the Approximation of the step function by some cumulative distribution functions, Compt. rend. Acad. bulg. Sci. 68(12) (2015) 1475–1482.

[25] V. Kyurkchiev, N. Kyurkchiev, On the Approximation of the Step function by Raised-Cosine and Laplace Cumulative Distribution Functions, European International Journal of Science and Technology 4(9) (2016) 75–84.

[26] A. Iliev, N. Kyurkchiev, S. Markov, Approximation of the Cut Function by Stannard and Richard Sigmoid Functions, International Journal of Pure and Applied Mathematics 109(1) (2016) 119–128.

[27] A. Blumberg, Logistic Growth Rate Functions, Journal of Theoretical Biology 21 (1968) 42–44.

[28] B. Gompertz, On the nature of the function expressive of the law of human mortality, Philosophical Transactions 27 (1825) 513–519.

[29] R. Pearl, The growth of populations, Quarterly Review of Biology 2 (1927) 532–548.